Principles of Communications

EES 351

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4.2 Energy and Power

Review: Energy and Power

- Consider a signal $g(t)$.
- Total (normalized) **energy**:

Parseval's Theorem [2.43]

[Defn. 4.13]

$$\boxed{E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt} = \lim_{T\to\infty} \int_{-T}^{T} |g(t)|^2 dt \overset{\downarrow}{=} \int_{-\infty}^{\infty} |G(f)|^2 df.$$

- Average (normalized) **power**:

[Defn. 4.15]

$$\boxed{P_g = \left\langle |g(t)|^2 \right\rangle} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 dt.$$

time-average operator
[Defn. 4.16a]

Review: Time average vs. Inner Product

Inner Product:

$$\left\langle x(t), y(t) \right\rangle = \int_{-\infty}^{\infty} x(t) y^{*}(t) dt$$

two arguments

Time Average:

$$\left\langle g(t) \right\rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t) dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} g(t) dt.$$

one argument

Power Calculation

	$g(t)$	$P_g = \langle \lvert g(t) \rvert^2 \rangle$
[4.18]	Periodic with period T_0	$\frac{1}{T_0} \int_{T_0} \lvert g(t) \rvert^2 dt$
Linear combination of complex exponential functions [4.23]	$\sum_k c_k e^{j2\pi f_k t}$ where the f_k are distinct	$\sum_k \lvert c_k \rvert^2$

Σ P_Σ

Power Calculation: Special Cases

	$g(t)$	$P_g = \langle \lvert g(t)\rvert^2 \rangle$
Linear combination of complex exponential functions [4.23]	$\sum_k c_k e^{j2\pi f_k t}$ where the f_k are distinct	$\sum_k \lvert c_k\rvert^2$
Linear combination of sinusoids [4.28]	$\sum_k A_k \cos(2\pi f_k t + \phi_k)$ where the f_k are positive and distinct	$\frac{1}{2}\sum_k \lvert A_k\rvert^2$

"$P_\Sigma = \Sigma P$"

Summary (1)

- **(Total) Energy:** $\boldsymbol{E_g} = \int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |G(f)|^2\, df$
- **Average Power:** $\boldsymbol{P_g} = \langle |g(t)|^2 \rangle = \lim_{T\to\infty} \boxed{\frac{1}{2T}\int_{-T}^{T} |g(t)|^2\, dt}$ “energy per unit time”
  - For periodic signal:
    $P_g = \frac{1}{T_0}\int_{T_0} |g(t)|^2\, dt = \frac{\text{energy in one period}}{\text{period}}$
  - Other special cases:

	$g(t)$	$P_g = \langle \lvert g(t)\rvert^2 \rangle$
Linear combination of complex exponential functions [4.23]	$\sum_k c_k e^{j2\pi f_k t}$ where the f_k are distinct	$\sum_k \lvert c_k\rvert^2$
Linear combination of sinusoids [4.28]	$\sum_k A_k \cos(2\pi f_k t + \phi_k)$ where the f_k are positive and distinct	$\frac{1}{2}\sum_k \lvert A_k\rvert^2$

- **Time Average:** $\langle \boldsymbol{g(t)} \rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} g(t)\, dt$
  - For periodic signal:
    $\langle g(t) \rangle = \frac{1}{T_0}\int_{T_0} g(t)\, dt$

Summary (2)

- **(Total) Energy:** $E_g = \int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |G(f)|^2\, df$
  - A signal $g(t)$ is an **energy signal** if $0 < E_g < \infty$.
    - Any energy signal $g(t)$ has $P_g = 0$.

“energy per unit time”

- **Average Power:** $P_g = \langle |g(t)|^2 \rangle = \lim_{T\to\infty} \boxed{\frac{1}{2T} \int_{-T}^{T} |g(t)|^2\, dt}$
  - A signal $g(t)$ is a **power signal** if $0 < P_g < \infty$.
    - Any power signal $g(t)$ has $E_g = \infty$.
  - For periodic signal:

    $$P_g = \frac{1}{T_0} \int_{T_0} |g(t)|^2\, dt = \frac{\text{energy in one period}}{\text{period}}$$
  - Other special cases:

	$g(t)$	$P_g = \langle \lvert g(t)\rvert^2 \rangle$
Linear combination of complex exponential functions [4.23]	$\sum_k c_k e^{j2\pi f_k t}$ where the f_k are distinct	$\sum_k \lvert c_k\rvert^2$
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Summary (3)

- **(Total) Energy:** $E_g = \int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |G(f)|^2\, df$
  - A signal $g(t)$ is an **energy signal** if $0 < E_g < \infty$.
    - Any energy signal $g(t)$ has $P_g = 0$.
- **Average Power:** $P_g = \langle |g(t)|^2 \rangle = \lim_{T \to \infty} \boxed{\frac{1}{2T} \int_{-T}^{T} |g(t)|^2\, dt}$

"energy per unit time"

Time Average:

$$\langle \boldsymbol{g(t)} \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} g(t)\, dt$$

For periodic signal:

$$\langle g(t) \rangle = \frac{1}{T_0} \int_{T_0} g(t)\, dt$$

  - A signal $g(t)$ is a **power signal** if $0 < P_g < \infty$.
    - Any power signal $g(t)$ has $E_g = \infty$.
  - For periodic signal:

$$P_g = \frac{1}{T_0} \int_{T_0} |g(t)|^2\, dt = \frac{\text{energy in one period}}{\text{period}}$$

  - Other special cases:

	$g(t)$	$P_g = \langle \lvert g(t) \rvert^2 \rangle$
Linear combination of complex exponential functions [4.23]	$\sum_k c_k e^{j2\pi f_k t}$ where the f_k are distinct	$\sum_k \lvert c_k \rvert^2$
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Power Calculation

$g(t)$	$P_g = \langle \lvert g(t) \rvert^2 \rangle$
Periodic with period T_0	$\frac{1}{T_0} \int_{T_0} \lvert g(t) \rvert^2 dt$
$\sum_k a_k(t)$ where the $a_k(t)$ are **orthogonal** (e.g., do not overlap in the frequency domain)	$\sum_k P_{a_k}$

Time average vs. Inner Product

Inner Product:

$$\left\langle x(t), y(t) \right\rangle = \int_{-\infty}^{\infty} x(t)\, y^{*}(t)\, dt$$

two arguments

Time Average:

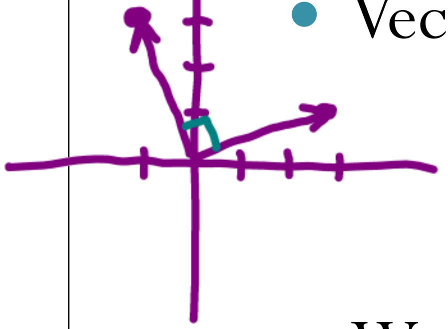
$$\left\langle g(t) \right\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t)\, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(t)\, dt.$$

one argument

Ex. $\left\langle \begin{pmatrix}-1\\3\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix} \right\rangle = (-1)(3)^* + (3)(1)^* = -3+3 = 0$

↳ The two vectors are orthogonal

Inner Product (Cross Correlation)



- Vectors

$$\langle \vec{x},\vec{y}\rangle = \vec{x}\cdot\vec{y}^* = \begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix}\cdot\begin{pmatrix}y_1\\ \vdots \\ y_n\end{pmatrix}^* = \sum_{k=1}^{n} x_k y_k^*$$

Complex conjugate

When the vectors are real-valued, the operation is the same as dot product that you have seen in high school.

- Waveforms: Time-Domain

[Defn. 4.15b] $$\langle x(t), y(t)\rangle = \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$$

By Parseval's Theorem [2.43], these two calculations will give the same answer.

- Waveforms: Frequency Domain

$$\langle X(f), Y(f)\rangle = \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, df$$

Inner Product (Cross Correlation)

- Vectors

Complex conjugate

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y}^* = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}^* = \sum_{k=1}^{n} x_k y_k^*$$

When the vectors are real-valued, the operation is the same as dot product that you have seen in high school.

Example:

$$\left\langle \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \right\rangle = (1)(-1) + (2)(0) + (-1)(-1) = 0$$

Orthogonality

- Two signals are said to be **orthogonal** if their **inner product** is zero.
- The symbol $\perp$ is used to denote orthogonality.

Vector:

$$\left\langle \vec{a},\vec{b} \right\rangle = \vec{a}\cdot\vec{b}^{*} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}^{*} = \sum_{k=1}^{n} a_k b_k^{*} = 0$$

Time-domain:

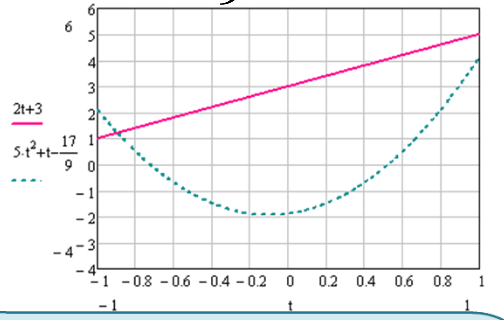
$$\left\langle a,b \right\rangle = \int_{-\infty}^{\infty} a(t)\,b^{*}(t)\,dt = 0$$

Frequency domain:

$$\left\langle A,B \right\rangle = \int_{-\infty}^{\infty} A(f)\,B^{*}(f)\,df = 0$$

Example:

$$2t+3 \text{ and } 5t^2+t-\frac{17}{9} \text{ on } [-1,1]$$



Example (Fourier Series):

$$\sin\left(2\pi k_1 \frac{t}{T}\right) \text{ and } \cos\left(2\pi k_2 \frac{t}{T}\right) \text{ on } [0,T]$$

$$e^{j2\pi n\frac{t}{T}} \text{ on } [0,T]$$

Important Properties

Multiple Access (MA)

① FDMA

freq. division

② TDMA

time

③ CDMA

code

- Parseval's theorem

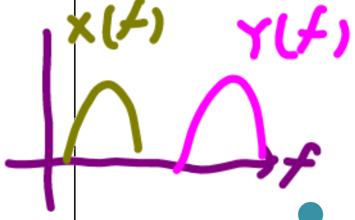
$$\langle x,y\rangle \equiv \int_{-\infty}^{\infty} x(t)\,y^{*}(t)\,dt = \int_{-\infty}^{\infty} X(f)\,Y^{*}(f)\,df \equiv \langle X,Y\rangle$$

It is therefore sufficient to check only on the "convenient" domain.

$$x(t)\perp y(t) \quad \text{iff} \quad X(f)\perp Y(f).$$

- Useful observation: If the non-zero regions of two signals
  - do not overlap in time domain or
  - do not overlap in frequency domain,

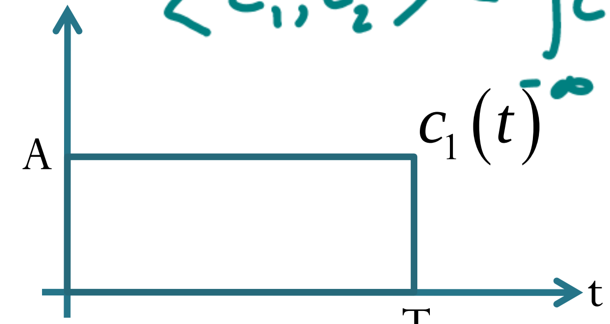
  then the two signals are orthogonal (their inner product = 0).

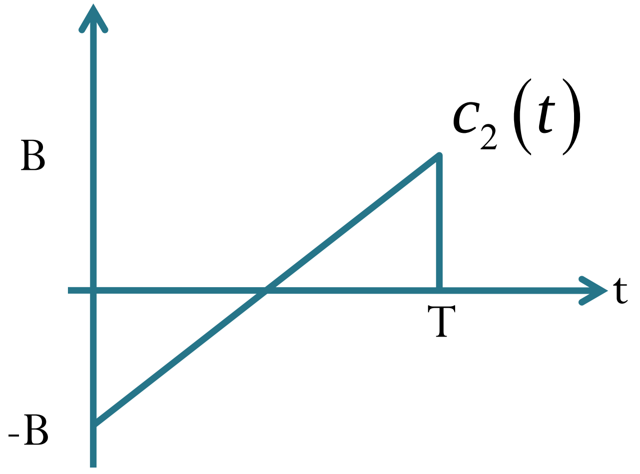


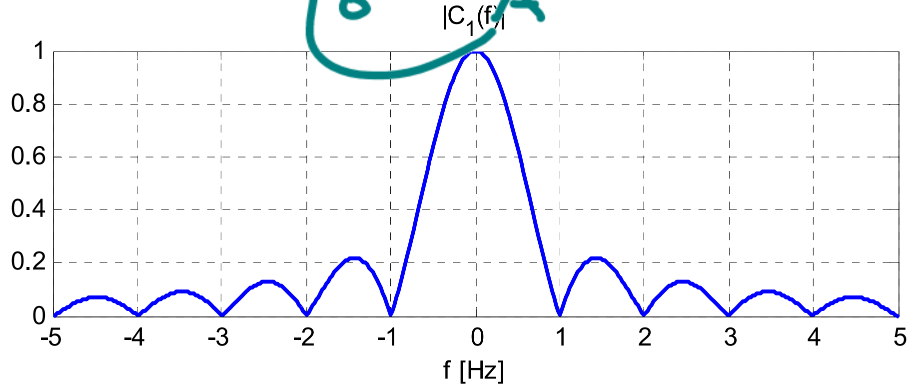
- However, in general, orthogonal signals may overlap both in time and in frequency domain.

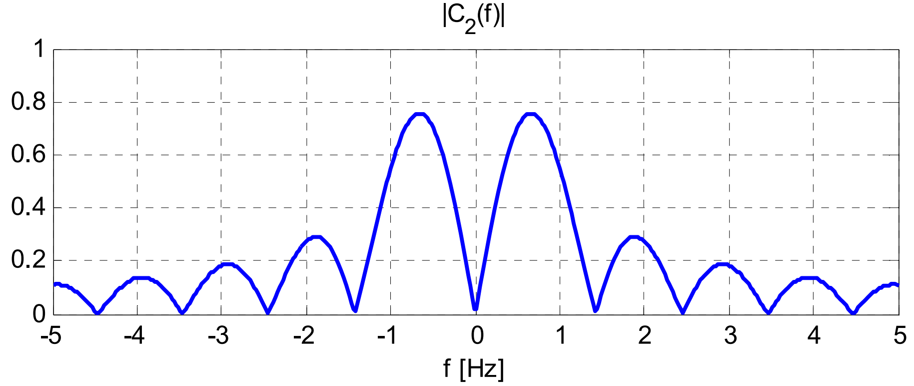
Orthogonality: Example 1

$$\langle c_1, c_2 \rangle = \int_{-\infty}^{\infty} c_1(t)\, c_2^*(t)\, dt = \int_0^T \underbrace{c_1(t)}_{A}\, c_2(t)\, dt = 0$$







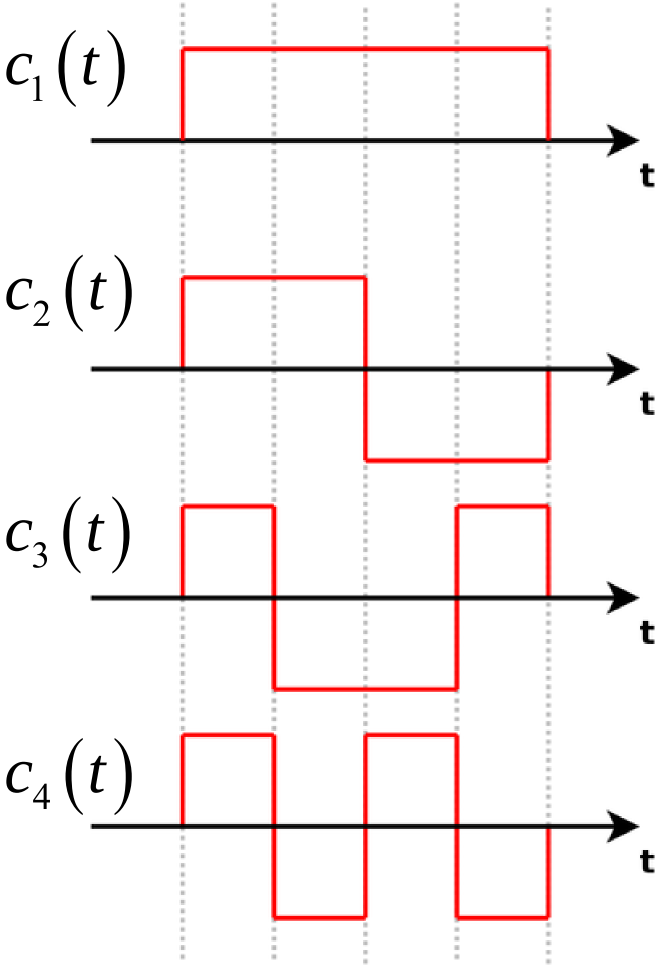


[CDMAEx.m]

The two waveforms above overlaps both in time domain and in frequency domian.

Orthogonality: Example 2

An example of four "mutually orthogonal" signals.



When $i \neq j$,

$$\langle c_i(t), c_j(t) \rangle = 0$$

Power Calculation

$g(t)$	$P_g = \langle \lvert g(t) \rvert^2 \rangle$
$\sum_k a_k(t)$ where the $a_k(t)$ are **orthogonal** (e.g., do not overlap in the frequency domain)	$\sum_k P_{a_k}$

orthogonal summands

$$P_\Sigma \overset{\downarrow}{=} \Sigma P$$

Special Cases: A Revisit

$g(t)$	$P_g = \langle \lvert g(t) \rvert^2 \rangle$
$\sum_k c_k e^{j2\pi f_k t}$ where the f_k are distinct	$\sum_k \lvert c_k \rvert^2$
$\sum_k A_k \cos(2\pi f_k t + \phi_k)$ where the f_k are positive and distinct	$\frac{1}{2} \sum_k \lvert A_k \rvert^2$

The requirement that "the f_k are distinct" is there to guarantee that summands do not overlap in the frequency domain. This makes them orthogonal.

Power Calculation

$g(t)$	$P_g = \langle \lvert g(t) \rvert^2 \rangle$
Periodic with period T_0	$\frac{1}{T_0} \int_{T_0} \lvert g(t) \rvert^2 dt$
$\sum_k a_k(t)$ where the $a_k(t)$ are **orthogonal** (e.g., do not overlap in the frequency domain)	$\sum_k P_{a_k}$